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SOME FINITE GROUPS WITH LARGE CONJUGACY CLASSES†

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ABSTRACT

We derive some properties of a family of finite groups, which was investigated by Camina, Macdonald, and others. For instance, we give information about the Schur multipliers of the class two p-groups in this family.

Introduction

We are interested in finite groups G, having a normal subgroup N, with the following property: an element x of G outside N is conjugate to all elements in its coset xN. E.g., G may be a Frobenius group with N the Frobenius kernel; or G may be extra special with N the centre. These groups were introduced by A. Camina [C], who has shown that if G is not Frobenius (with N the Frobenius kernel), then either N or G/N is a p-group. They have since attracted the attention of several mathematicians, and occur naturally in some other contexts.

It seems natural to divide these groups into four types, as follows:

(I) Frobenius groups, with N the Frobenius kernel.

(II) p-groups.

(III) Not of type (i) or (ii), and N is a p-group.

(IV) Not of type (i) or (ii), and G/N is a p-group.

We refer to groups of types (ii), (iii), (iv) as *Camina groups*, and N is called the *Camina kernel*.

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In this paper we discuss groups of types (ii) and (iv). First, when G is a p-group, N must be a term of the lower central series [McD1], and the most interesting case is $N = G'$. The main results for this case appear in [McD1], [McD2], [MS], and here we give some results about the power structure of these groups (these results are needed in [MS]) and, by generalising slightly some arguments from [McD1], we are able to get information about the Schur multiplier of Camina groups of nilpotency class two and prove some related results. In many cases, the Schur multiplier is explicitly determined (see Theorem 5). We remark that Camina groups of class two are also known as semi-extra-special groups (see e.g. [Be]).

Now let G be of type (iv), let P be a Sylow p-subgroup of G, and let $K = P \cap N$. Then K is a term of the upper central series of P [CM]. The case $K = Z(P)$ was discussed in [CMS], and we show here that $K = Z_2(P)$ is impossible, at least for odd p. We also derive a curious property of the factor group $Q = G/N$, namely: any two maximal cyclic subgroups of Q with a non-trivial intersection (for $p = 2$: with intersection of size 4 at last) are conjugate. We intend to investigate p-groups with this property in a later paper.

Non-standard notation and terminology will usually be introduced when they first occur. We use $\Pi(G)$ and $\Omega(G)$ for the subgroups generated by the pth powers and the elements of order p of the p-group G, respectively. The term *unicentre*, introduced at the beginning of section 2, may be of interest.

1. THEOREM 1. *Let G be a p-group of class* 3, *having a central subgroup M, such that $M \leq G'$ and G/M is a Camina group of class 2 (the Camina kernel is then necessarily G'/M). Denote $|G:G'| = p^m$, $|G':M| = p^n$. Then:*

(a) *$M = Z(G)$, $m = 2n$, and $|G_3| \leq p^{2n}$.*

(b) *If also $M = G_3$, then either G/G_3 has exponent p (and p is odd), or G is dihedral, quaternion, or semi-dihedral of order* 16.

(c) *If* $\exp(G/M) > p$, *and G is not one of the exceptions of order 16, then $G_3 = \Pi(M) = \Pi(G')$, and $G' = M \times E$, where E is elementary abelian of order p^n.*

PROOF. The proof of part (a) is almost the same as the proof of Theorem 5.2 of [McD1], of which our result is a slight generalisation (in [McD1], where the assumption is that G itself is a Camina group, it is also shown that n is even, which does not hold here). Still, we give the proof in full, since we have found out, by bitter experience, that not to do so can be very confusing.

By assumption $G_3 \leq M$. Dividing out a maximal subgroup of G_3, we may assume that $|G_3| = p$. Then, if $x \in G'$, then $|G:C_G(x)| \leq p$. Denote $C =$

$C_G(G')$. Letting $G' = \langle M, x_1, \ldots, x_n \rangle$, we see that $C = C_G(x_1, \ldots, x_n)$, and hence $|G:C| \le p^n$. Let $c \in C - G'$, let b centralize c (modulo M), and let $x \in G$. Then $[b,c,x] = 1$ and $[x,b,c] = 1$, so $[c,x,b] = 1$. By assumption, the commutators $[c,x]M$ generate G'/M, and M is central, so b centralizes G', i.e. $b \in C$. Now let $a \in G - C$, and denote $A = C_G$ (a mod M). The remark just made shows that $A \cap C = G'$, and G/M being a Camina group shows that $|G:A| = p^n$. Therefore $|C:G'| = |C:C \cap A| \le |G:A| = p^n$, and so $|G:G'| \le p^{2n}$. But also $|G:G'| \ge p^{2n}$, by Theorem 3.2 of [McD1], so $m = 2n$.

Since G/M is a Camina group, we have $Z(G/M) = G'/M$, and thus $Z(G) \le G'$. The equality $m = 2n$ shows that also $|G:C| = p^n$, and $|G:C_G(x_i)| = p$. As x_i can be any element of $G' - M$, we see that such elements are not central, and $Z(G) = M$.

This proves that $m = 2n$ and that $Z(G) = M$ when $|G_3| = p$, which obviously implies these equalities for the general case. For the last assertion in (a), where we cannot assume anything about $|G_3|$, let $x \in G' - M$, and let K be a maximal subgroup of G_3. Then the case $|G_3| = p$ shows that $xK \notin Z(G/K) = M/K$, so $[x,G] \not\le K$. Therefore $[x,G] = G_3$, and $|G_3| = |G:C_G(x)| \le |G:G'| = p^{2n}$.

The proof of (b) is a variation on similar proofs in [M1], [M2], [M4]. Assume again that $|G_3| = p$, and let a and A be as above. As a does not centralize G', $C_G(a) \ne A$, but all conjugates of a in A lie in aM, so $|A:C_G(a)| = p$, and also $|G':G' \cap C_G(a)| = p$. It follows that $|G:C_G(a)| = p^{n+1} = |C:C \cap C_G(a)|$, and therefore $G = C_G(a)C$. Denote $Z(a) = Z(C_G(a))$. Then

$$Z(a) \cap G' \le Z(C_G(a)) \cap Z(C) \le Z(G),$$

so $Z(a) \cap C = Z(a) \cap G' = G_3$. Let b be another element of $G - C$, and let $c \in Z(a) \cap Z(b)$. If $c \notin G_3$, then also $c \notin C$, and $C_G(a)$, $C_G(b)$, $C_G(c)$ all have the same index p^{n+1} in G, while the first two subgroups are contained in the third, so all three are equal, and $Z(a) = Z(b)$. Thus the subgroups $Z(a)/G_3$ (for various a's) and C/G_3 constitute a partition of $H = G/G_3$.

Now assume exp $H > p$. By well-known properties of partitions, all elements of order greater than p lie in the same constituent of the partition, which also contains $Z(H)$ [Ba, Zusatz 5.6]. In our case this constituent must be C/G_3. If p is odd we get an immediate contradiction, because the class two group H is regular, and so generated by its elements of maximal order. For $p = 2$, it is both well known and easy to verify that the "Hughes subgroup" of H, generated by the elements of order greater than 2, is of index 2 (or 1), so we have $|G:C| = 2$, i.e. $n = 1$, and $|H| = 8$. Since the quaternion group is not the central factor group of any group, H is the dihedral group of order 8, and this implies that

G itself is either dihedral, quaternion, or semi-dihedral [H, III.11.9]. Since G is of class three, we obtain $|G| = 16$ (even without assuming $|G_3| = p$).

Next, if $\exp H = p$, then p is odd, because groups of exponent 2 are abelian.

To prove (c), note first that $\exp(G/G') = p$, by [McD1], so also $\exp(M/G_3) = \exp(G'/G_3) = \exp(G_3) = p$. If some maximal subgroup K of M does not contain G_3, then G/K satisfies the assumptions of (b), so G/K is one of the three exceptions, which shows that G/G_3 is dihedral of order 8, and again G itself is one of the three exceptions, a contradiction. Thus $K \supseteq G_3$, and $G_3 = \Phi(M) = \Pi(M) = \Pi(G')$. Then $G' = M \times E$, where E is any complement of G_3 in $\Omega(G')$.

Remark. If $p = 2$, then neither of the three candidates for G is itself a Camina group with kernel G'. This gives a new proof to Theorem 3.1 of [McD2], which states that if G is a 2-group, which is a Camina group with kernel G', then G has class 2. Conversely, assuming this result, the proof of (b) can be somewhat simplified, at least if G itself is a Camina group.

Corollary 2. *Let the p-group G be a Camina group with respect to G′, and of class greater than 2. Then* G_i/G_{i+2} *has exponent p.*

The proof of this is standard. We can use, e.g., [M3, Theorem 4].

Theorem 3. *Let the p-group G be a Camina group with kernel G′, and let G have class greater than* 2.

(a) *If* $p = 3$, *then* $\Pi(G) = G_3$.

(b) *If* $p \geq 5$, *and* $G_4 \neq 1$, *then* $\Pi(G) \leq G_4$. *If* $G_5 \neq 1$, *then* $\Pi(G) \leq G_5$.

Proof. We already know that $\Pi(G) \leq G_3$. To prove (a), we may assume that $\Pi(G) = 1$, and try to show $G_3 = 1$. Then G has exponent 3, and therefore each element a commutes with all its commutators [H, III.6.6]. But if a is not in G', then these commutators comprise all of G', so G' centralizes a. Letting a vary over all elements of $G - G'$, we get $G' \leq Z(G)$, and so $G_3 = 1$.

(b) For the first statement, we may assume $|G_4| = p$. By Lemma 2.1 of [McD2], $|G_3| \leq p^{n+1}$, where we use the notations of Theorem 1 (actually, by 2.1 of [MS], equality holds), and where n is even, by [McD1]. Thus $|\Pi(G)| \leq p^{n+1}$. But G is now a regular p-group, so we have

$$|G:\Omega(G)| \leq p^{n+1} < p^{2n} = |G:G'|,$$

and therefore there exists an element a of order p outside G'. But then all the commutators of a, i.e. all elements of G', are also of order p. In the regular group

G we now have $\exp G' = p$, which is equivalent to $\Pi(G) \leq Z(G)$. But, by 2.1 of [MS], $Z(G) = G_4$.

For the proof of the last statement, we may assume that $|G_5| = p$, and repeat the same argument, using the facts, proved in 3.2 of [MS], that now $|G_4 : G_5| = p^n$ and $Z(G) = G_5$.

The reader will have noticed that, for these values of p, the same argument can be used to give a simpler proof for Theorem 1(b).

2. We now pass to Schur multipliers and central extensions. Let G be a p-group, and let H be a representation group of G. Thus, H has a normal subgroup $M \leq Z(H) \cap H'$, with $M \cong M(G)$, the Schur multiplier of G, and $H/M \cong G$. The central subgroup of G, $Z^*(G) = Z(H)/M$, is independent of H [BFS, 3.2], and G is called *unicentral* if $Z^*(G) = Z(G)$. In view of this, it seems natural to call $Z^*(G)$ the *unicentre* of G. By [BFS, 2.3], $Z^*(G) = 1$ if and only if G is *capable*, i.e. G is the central factor group of some group.†

Theorem 4. *Let G be a Camina group of class 2, and denote $|G : G'| = p^m$, $|G'| = p^n$. Then G is either unicentral or capable. Both possibilities occur. If G is capable, then $m = 2n$.*

Proof. Let H be a representation group of G. If H is of class 2, then G is unicentral. Let H have class 3. Then H and M satisfy the conditions of Theorem 1, which shows that $Z(H) = M$, and thus G is capable, and that $m = 2n$. To see that both possibilities occur, it suffices to consider groups of order p^3.

Theorem 5. *Let G be as in Theorem 4.*

(a) *If G is unicentral, then $M(G)$ is elementary abelian of order $p^{1/2m(m-1)-n}$.*

(b) *If G is capable, then $p^{2n^2-2n+1} \leq |M(G)| \leq p^{2n^2}$.*

Proof. We employ the following exact sequence, described in [BFS, section 4], where N is any central subgroup of the group G:

$$(G/G') \otimes N \to M(G) \to M(G/N) \to G' \cap N \to 1.$$

The map $M(G) \to M(G/N)$ is injective exactly when $N \leq Z^*(G)$ [BFS, 4.2]. In case (a), take $N = Z(G) = Z^*(G)$. Then G/N is elementary abelian of order p^m, so $M(G/N)$ is elementary abelian of order $p^{m(m-1)/2}$, and $G' \cap N = N$, which yields our result. In case (b), we have $m = 2n$. Take N to be a normal subgroup

†The idea of considering unicentrality and capability of Camina groups appeared first in a preliminary version of [McD2].

of order p. Then G/N is unicentral, by Theorem 4, so $|M(G/N)| = p^{2n(n-1)+1}$, by part (a), and $N \cap G' = N$, so $M(G)$ maps onto (a) subgroup of order $p^{2n(n-1)}$ of $M(G/N)$. This map is not injective, since $Z^*(G) = 1$, so its kernel has order at least p and at most $|(G/G') \otimes N| = p^{2n}$, which proves both inequalities.

For the simplest case, $n = 1$, Theorem 5(b) shows that $|M(G)|$ is p or p^2. It is well known that in this case the order is p^2 for odd p and p for $p = 2$, the relevant groups G being the non-abelian exponent p group of order p^3, or the dihedral group of order 8, respectively.

3. We now pass to Camina groups of type (iv) of the introduction. Thus, the kernel N is not a p-group, but G/N is. The only known examples of such groups are Frobenius groups with the quaternion group of order 8 as the complement, where we take the Camina kernel to be the unique subgroup containing the Frobenius kernel with index 2. Various conditions under which a Camina group of type (iv) must be one of these examples are given in [CMS]. In any case, it is known that such groups are p-nilpotent and soluble [CMS, Corollary 2.1].

Let P be a Sylow p-subgroup of G, let $K = P \cap N$, and let R be the normal p-complement of G. Then P acts on R, and every element of P that has some non-trivial fixed point in this action is contained in K [C]. In such circumstances the group P/K is called a *Frobenius Wielandt complement* (FW-complement — see e.g. [S]). We first give a result about such groups.

LEMMA 6. *If P/K is an FW-complement, then, for each $a \notin K$, $C_G(a)K/K$ is cyclic or quaternion.*

This is a corollary of Lemma 1.4 of [S].

LEMMA 7. *Let P/K be an FW-complement, let $H \leq K$, and assume that $H \geq C_K(H)$. Let $C = C_P(H)$. Then $C/Z(H)$ is elementary abelian, cyclic, dihedral, quaternion, or non-abelian of exponent p and order p^3.*

PROOF. Note that $K \cap C = Z(H) \leq Z(C)$. Assume that $C \neq Z(H)$, and let A be a maximal abelian subgroup of C. By Lemma 6, $A/Z(H)$ is cyclic. If $Z(C) \neq Z(H)$, we take an element $a \in Z(C) - Z(H)$, and Lemma 6 yields our claim. If $Z(C) = Z(H)$, then the structure of $C/Z(C)$ is given by Proposition 1 of [M2].

We now return to Camina groups. Let again G be a Camina group with kernel N, where G/N, but not G, is a p-group. Let P be a Sylow p-subgroup of G, and let $K = N \cap P$. By [CM], K is a member of each central series of P, so $K = Z_i(P)$, for some i. It is shown in Theorem 3 of [CMS] that if $i = 1$, then G is among the examples described above, i.e. $p = 2$, G is a Frobenius group with the quaternion group of order 8 as complement, and N contains the Frobenius kernel with index 2. We now show that the same conclusion holds if $i = 2$ and p is odd, which means that this case is impossible.

LEMMA 8. *With the same assumptions, let* $K = Z_i(P)$, *and let* $\mathrm{cl}(P) = c$. *Then* $c \leq 2i + 1$.

PROOF. Since K is also a member of the lower central series of P, we have $K = P_{c+1-i}$. Suppose $c \geq 2i + 2$. Then $K \leq P_{c-i} \leq Z_{i+2}(P) \leq Z_{c-i}(P)$. Let $a \in P_{c-i} - K$. Then $Z_{i+2}(P) \leq C_P(a)$, so, by Lemma 6, $Z_{i+2}(P)/Z_i(P) = Z_2(P/Z_i(P))$ is cyclic or quaternion. In the first case $P/Z_i(P)$ itself is cyclic, or contains a cyclic subgroup of index 2 [H, III.7.7]. But the only capable group among these is the dihedral one, and if $P/Z_i(P)$ is dihedral then P itself is quaternion, dihedral or semi-dihedral, contradicting Theorem 3 of [CMS]. The second case is eliminated similarly, applying the following lemma, whose proof we leave to the reader.

LEMMA 9. *Let* P *be a 2-group, such that* $Z_2(P)$ *is quaternion. Then* P *is the quaternion group of order 8.*

THEOREM 10. *With the same assumptions and notations as above, and also assuming that* p *is odd, we cannot have* $K = Z_2(P)$.

PROOF. By Lemma 8, we have $c \leq 5$, so by [CMS, Theorem 3 (4)(7)] we may assume that $c = 5$ and that $p = 3$. We denote $Q = P/K$.

Thus $K = P_4 = Z_2$ (we write Z_i for $Z_i(P)$). Applying Lemma 6 to an element of $P_3 - K$, we see that Z_3/K is cyclic, while [McD1] shows that the factors Z_{i+1}/Z_i, for $i > 1$, are elementary. Therefore $|Z_3 : K| = p$, and $Z_3 = P_3$. Next, apply Lemma 7, with $H = K$, to get the possible structures for Z_4/K. If it is cyclic, we obtain the same contradiction as in the proof of Lemma 8. If it is non-abelian, it must have order p^3, so $|Z_4 : Z_3| = p^2$, P' lies in Z_4 with index 1 or p, and P' centralizes Z_4 (mod K), so P'/K is central in Z_4/K, which is abelian, after all. Thus Z_4/K is elementary abelian, and Q has exponent p or p^2.

Let a be a fixed element in $Z_3 - K$. Then $C_P(a) \supseteq \langle a, K \rangle$, $C_P(a)/K$ is cyclic, so $|C_P(a)| = 3|K|$ or $9|K|$, and also $|C_P(a)| = |C_Q(aK)| = |Q|$. Now let b be an element of $Z_4 - Z_3$. Then again $|C_P(b)|$ is $3|K|$ or $9|K|$, and $|C_P(b)| = |C_Q(bK)| < |Q|$. Thus we must have $|C_P(a)| = 9|K|$ and $|C_P(b)| = 3|K|$, so $C_P(b) = \langle b, K \rangle$ and $C_P(a) = \langle c, K \rangle$, where a and c can be chosen so that $a =$

c^3. On the other hand, the form of $C_P(b)$ shows that b is not the third power of any element. Thus $\Pi(Q) = Z_3/K$.

Consider any element of order 9 in Q. By Proposition 11 below, this element is conjugate to cK or $c^{-1}K$. Then all these elements are contained in $\langle cK, Q' \rangle$. On the other hand, they generate a subgroup of index ≤ 3 of Q, and thus $|Q:Q'| = 9$, and so $|Q':Q_3| = 3$, and $Q = P/P_4$ is a group of order 3^4 and class 3. Then P itself is of maximal class, and this case was discussed in [CMS, Theorem 3].

Proposition 11. *Let G be a Camina group with kernel N, with G/N, but not G, a p-group. Assume that p is odd. Then any two maximal cyclic subgroups of $Q = G/N$, which have a non-trivial intersection, are conjugate. The same conclusion holds for $p = 2$, provided the intersection has size 4 at least.*

Proof. Let P be a Sylow p-subgroup of G, and let $K = P \cap N$. Then we can take $Q = P/K$. Let C/K and D/K be two intersecting maximal cyclic subgroups of Q. We choose elements c and d of P, generating C and D (mod K), and let a be a power of c generating (mod K) the subgroup of order p (order 4 if $p = 2$) in C/K. Then d has a power in aK, say ak. Lemma 6 shows that $C_P(a)K/K$ and $C_P(ak)K/K$ are cyclic, so the maximality implies that these subgroups are C/K and D/K, respectively. Thus C and D are conjugate, as a and ak are conjugate.

Obvious examples of groups satisfying this conjugacy property are groups of exponent p, and cyclic and dihedral groups. As a less trivial example, let P be a group of order 3^4 and class 3. Then P contains a unique abelian maximal subgroup, P_1 say, and P_1 is either elementary or of type (9,3). Suppose the latter, and also suppose that all elements outside P_1 are of order 3. Then P contains 3 cyclic subgroups of order 9, which are all contained in P_1, and are conjugate. Thus P satisfies our property. There exists exactly one group of order 3^4 satisfying all these assumptions, and, except for the trivial examples noted above, it is the only group of order p^4 with the conjugacy property of Proposition 11. For each $p > 3$, however, there exist examples of orders p^5 and p^6. We will return to these groups in a later paper.

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